

Closing Tue: Taylor Notes 1, 2
 Closing Thu: Taylor Notes 3
 Final: Sat, June 2, 5:00pm, KANE 130
 Eight pages, covers everything.

TN 2 & 3: Higher order approx.

Recall: 1st Taylor polynomial

$$T_1(x) = f(b) + f'(b)(x - b)$$

Error Bound

On interval $[a, b]$, if $|f'''(x)| \leq M$,
 then $|f(x) - T_1(x)| \leq \frac{M}{2} |x - b|^2$.

Entry Task: Let $f(x) = x^{1/3}$.

- Find the 1st Taylor Polynomial based at $b = 8$.
- Give a bound on the error over the interval $[7, 9]$.

$$f(x) = x^{1/3} \rightarrow f(8) = 2$$

$$f'(x) = \frac{1}{3} x^{-2/3} = \frac{1}{3 x^{2/3}} \rightarrow f'(8) = \frac{1}{3 \cdot 4} = \frac{1}{12}$$

$$f''(x) = -\frac{2}{9} x^{-5/3} = -\frac{2}{9 x^{5/3}}$$

$$T_1(x) = 2 + \frac{1}{12}(x - 8)$$

THUS, $x^{1/3} \approx 2 + \frac{1}{12}(x - 8)$ for x CLOSE TO 8.

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 Error Bound \swarrow DECREASING ON $[7, 9]$

$$|f''(x)| = \frac{2}{9 x^{5/3}} \leq \frac{2}{9(7)^{5/3}} \approx 0.008675 = M$$

$$\Rightarrow \text{Error} \leq \frac{0.008675}{2} |x - 8|^2$$

$$\approx 0.0043377 \leftarrow \text{Error Bound}$$

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EXAMPLE $\sqrt[3]{9} \approx 2 + \frac{1}{12}(9 - 8)$
 $= 2 + \frac{1}{12} = 2.08\bar{3} \pm 0.004$

"ACTUAL" = 2.0800838...

2nd Taylor Polynomial is given by

$$T_2(x) = f(b) + f'(b)(x - b) + \frac{1}{2}f''(b)(x - b)^2$$

Quadratic error bound theorem

On interval $[a, b]$, if $|f'''(x)| \leq M$,
then $|f(x) - T_2(x)| \leq \frac{M}{6}|x - b|^3$.

Example:

Find the 2nd Taylor polynomial for $f(x) = x^{1/3}$ based at $b = 8$ and find an error bound on the interval $[7, 9]$.

$$f''(8) = -\frac{2}{9(8)^{5/3}} = -\frac{2}{9 \cdot 32} = -\frac{1}{144}$$

$$T_2(x) = 2 + \frac{1}{12}(x - 8) + \frac{1}{2}\left(-\frac{1}{144}\right)(x - 8)^2$$

$$= 2 + \frac{1}{12}(x - 8) - \frac{1}{288}(x - 8)^2$$

$$x^{1/3} \approx \underbrace{2 + \frac{1}{12}(x - 8) - \frac{1}{288}(x - 8)^2}$$

for x CLOSE TO 8

ERROR BOUND

$$f'''(x) = \frac{10}{27}x^{-8/3} = \frac{10}{27x^{8/3}}$$

$$|f'''(x)| \leq \frac{10}{27(7)^{8/3}} \approx 0.002065577$$

$$\begin{aligned} \text{Error} &\leq \frac{0.0020656}{6} |x - 8|^3 \quad \leftarrow 9 \text{ (or 7)} \\ &= 0.000344 \end{aligned}$$

EXAMPLES

$$\sqrt[3]{9} \approx 2 + \frac{1}{12}(9 - 8) - \frac{1}{288}(9 - 8)^2$$

$$\approx \boxed{2.079867} \pm 0.000344$$

"ACTUAL" = 2.0800838

Taylor Approximation Idea:

If two functions have **all** the same derivative values, then they are the same function (up to a constant).

Let's compare derivatives of $f(x)$ and $T_2(x)$ at b .

$$\begin{aligned} T_2(x) &= f(b) + f'(b)(x-b) + \frac{1}{2}f''(b)(x-b)^2 + \frac{1}{3!}f'''(b)(x-b)^3 \\ T_2'(x) &= 0 + f'(b) + f''(b)(x-b) \leftarrow \text{CANCEL} \\ T_2''(x) &= 0 + 0 + f''(b) \\ T_2'''(x) &= 0 \end{aligned}$$
$$\begin{aligned} &+ \frac{1}{3!}f'''(b)(x-b)^3 \\ &+ \frac{1}{3}f'''(b)3(x-b)^2 \\ &+ \frac{1}{2} \frac{1}{3} f'''(b) 3 \cdot 2 (x-b) \\ &\quad \uparrow \\ &\quad 6 \end{aligned}$$

Now plug in $x = b$ to each of these.

- What do you see?
- Why did we need a $\frac{1}{2}$?
- What would $T_3(x)$ look like?
- What would $T_4(x)$ look like?
($T_5(x)$?, $T_6(x)$?...)

$$\begin{aligned} &+ \frac{1}{3!} f'''(b)(x-b)^3 \\ &+ \frac{1}{4!} f^{(4)}(b)(x-b)^4 \\ &+ \frac{1}{5!} f^{(5)}(b)(x-b)^5 \end{aligned}$$

$$\begin{aligned} 3! &= 3 \cdot 2 \cdot 1 = 6 \\ 4! &= 4 \cdot 3 \cdot 2 \cdot 1 = 24 \\ 5! &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \\ 0! &= 1 \leftarrow \text{DEFIN} \\ 1! &= 1 \\ 2! &= 2 \end{aligned}$$

n^{th} Taylor polynomial

$$f(b) + f'(b)(x-b) + \frac{1}{2}f''(b)(x-b)^2 + \frac{1}{3!}f'''(b)(x-b)^3 + \dots + \frac{1}{n!}f^{(n)}(b)(x-b)^n$$

In sigma notation:

$$T_n(x) = \sum_{k=0}^n \frac{1}{k!} f^{(k)}(b)(x-b)^k$$

Example: Find the 9th Taylor polynomial for $f(x) = e^x$ based at $b = 0$, and give an error bound on the interval $[-2, 2]$.

$$f(x) = e^x \rightarrow f(0) = 1$$

$$f'(x) = e^x \rightarrow f'(0) = 1$$

$$f''(x) = e^x \rightarrow f''(0) = 1$$

$$f^{(9)}(x) = e^x \rightarrow f^{(9)}(0) = 1$$

$$1 + 1(x-0) + \frac{1}{2!}1(x-0)^2 + \frac{1}{3!}1(x-0)^3 + \dots + \frac{1}{9!}1(x-0)^9$$

Taylor's Inequality (error bound):

on a given interval $[a, b]$,

if $|f^{(n+1)}(x)| \leq M$, then

$$|f(x) - T_n(x)| \leq \frac{M}{(n+1)!} |x-b|^{n+1}$$

$$e^x \approx 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots + \frac{1}{9!}x^9$$

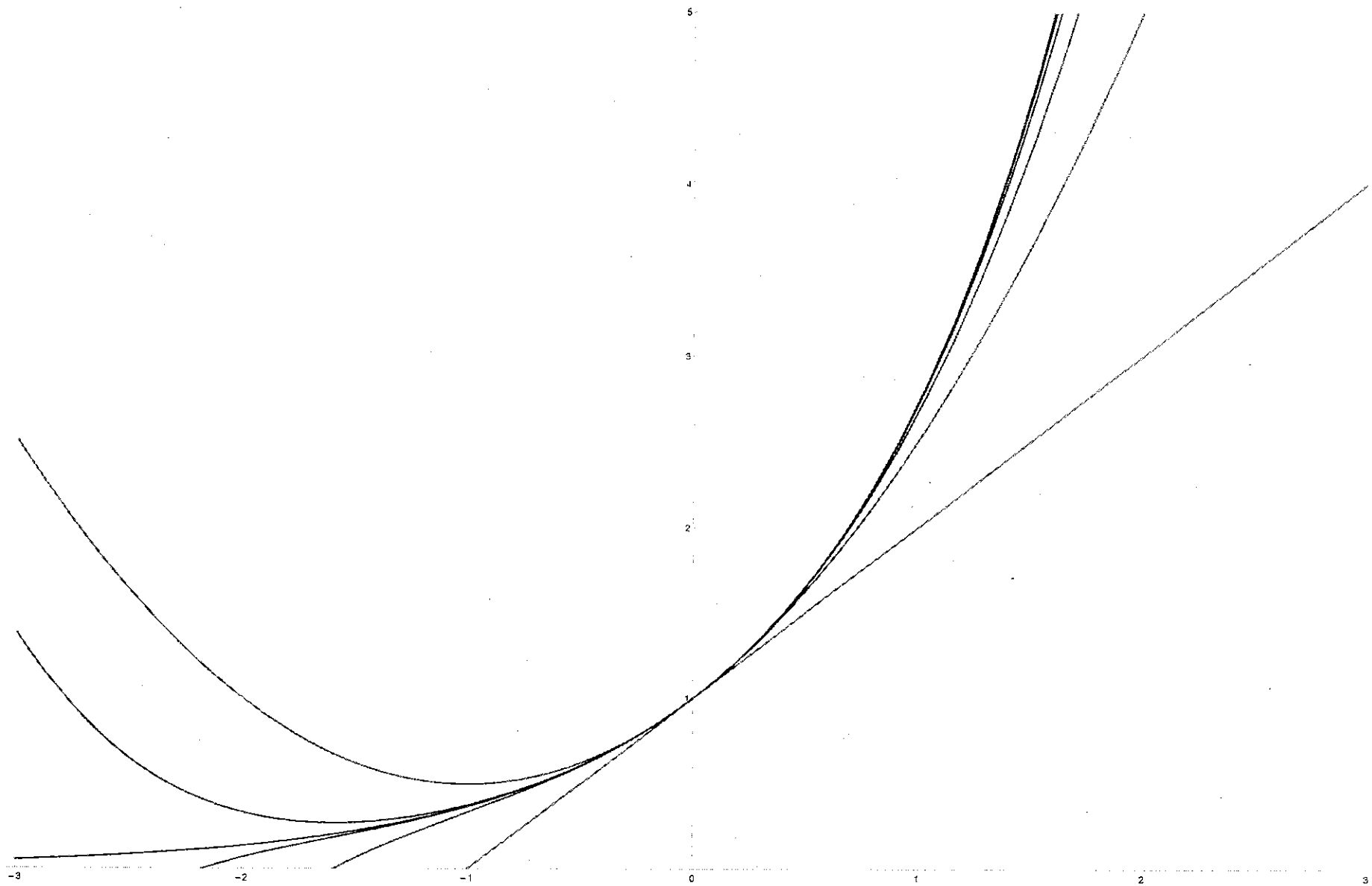
Error Bound \swarrow INCREASING FUNCTION
MAX AT $x=2$
 $f^{(10)}(x) = e^x \leq e^2 = M$

$$\text{Error} \leq \frac{e^2}{10!} |x-0|^{10} \leq \frac{e^2}{10!} 2^{10} \approx 0.002095$$

EXAMPLE

$$e^{1.8} \approx 1 + (1.8) + \frac{1}{2!}(1.8)^2 + \dots + \frac{1}{9!}(1.8)^9 \pm 0.002$$

$f(x) = e^x$ and
 $T_1(x), T_2(x), T_3(x), T_4(x), T_5(x)$



Example: Again consider,

$$f(x) = e^x \text{ based at } b = 0$$

Find the first value of n when

Taylor's inequality gives an error

less than 0.0001

on $[-2, 2]$.

$$\text{WANT } \frac{e^2}{(n+1)!} |x-0|^{n+1} \leq 0.0001$$

THERE IS NO GOOD WAY TO SOLVE THIS EXACTLY.

SO WE GUESS AND CHECK!

$n=9 \Rightarrow$ NO (WE JUST DID IT AND ERROR WAS 0.004)

$n=10 \Rightarrow$ Error $\leq \frac{e^2}{11!} 2^{11} \approx 0.000379$ NOT SMALL ENOUGH YET

$n=11 \Rightarrow$ Error $\leq \frac{e^2}{12!} 2^{12} \approx 0.000063$ YES!

$$n=11$$

Side Note:

For a fixed constant, a , the expression $\frac{a^k}{k!}$ goes to zero as k goes to infinity.

So the expression $\frac{1}{(n+1)!} |x - b|^{n+1}$, will always go to zero as n gets bigger.

Which means that the error goes to zero, unless something unusual is happening with M , which will see in examples later.

TN 4: Taylor Series

Def'n: The **Taylor Series** for $f(x)$ based at b is defined by

$$\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(b)(x-b)^k = \lim_{n \rightarrow \infty} T_n(x)$$

If the limit exists at x ,
then we say it **converges** at x .
(i.e. the error goes to zero at x)

Otherwise, we say it **diverges** at x .

The **open interval of convergence** gives the largest open interval over which the series converges.

Note: If

$$\lim_{n \rightarrow \infty} \frac{M}{(n+1)!} |x-b|^{n+1} = 0$$

then x is in the open interval of convergence.